Chapter 8: Linear Feedback Systems

Spring 2009/10
Lecture: Tim Woo
Where we are

Continuous-time

Hardware Implementation ➔ Closed-loop Systems ➔ State-space model ➔ Differential equations ➔ System Characteristics ➔ System Responses ➔ CTFT ➔ Laplace Transform

Open-loop Systems ➔ State-space model ➔ Mapping ➔ Discrete-time

Closed-loop Systems ➔ Hardware Implementation ➔ State-space model ➔ Difference equations ➔ System Characteristics ➔ System Responses ➔ DTFT ➔ z-Transform

Open-loop Systems

Will be covered if available

Done in 211

To be covered

In progress

Done
Expected Outcome

• In this chapter, you will be able to
  – Understand the importance of linear feedback systems
  – Analogy the applications of linear feedback systems
  – Introduce and compare two methodologies on the evaluation of stability of feedback systems
    • the root-locus analysis
    • Nyquist stability criterion
  – Examine the conditions of stability of feedback systems by applying
    • the root-locus analysis
    • Nyquist stability criterion
  – Introduce and examine the concept of the margin of stability (gain and phase margins) in feedback system.
Outline

- Section 11.0 Introduction
- Section 11.1 Linear Feedback Systems
- Section 11.2 Some applications and consequence of feedback
- Section 11.3 Root-locus analysis of linear feedback systems
- Section 11.4 The Nyquist stability criterion
- Section 11.5 Gain and phase margins
Section 11.0 Introduction

- So far, the systems we learnt are referred as open-loop systems. The output of a system is determined by the characteristics of the input signal.

Example:

![Diagram of a system with input voltage and output angular position.](image)
Section 11.0 Introduction

• Instead, we could suggest a different method for pointing the telescope - the feedback system, by using the output of a system to control or modify the input.

• A feedback system is referred as Closed-loop system.
Section 11.0 Introduction

• The closed-loop system have two important advantages:
  – Provide an error-correcting mechanism that can reduce sensitivity to the disturbances and to errors in the modeling of the system
  – Stabilize a system that is inherently unstable

• Typical applications
  – Chemical process control
  – Automotive fuel systems
  – Household heating systems
  – Aerospace systems
  – Stabilizing an inverted pendulum, etc

![An Inverted Pendulum](image)
Section 11.1 Linear Feedback Systems

- The basic feedback system is configured in

\[ H(s) \] (or \[ H(z) \]) is referred as the system function in the forward path

- \[ G(s) \] (or \[ G(z) \]) is referred as the system function in the feedback path
Section 11.1 Linear Feedback Systems

• From the diagram, we obtain the relation

\[ Y(s) = H(s)E(s) \]
\[ E(s) = X(s) - R(s) \]
\[ R(s) = G(s)Y(s) \]

• These give the closed-loop system function is

\[ Y(s) = H(s)E(s) \]
\[ = H(s)[X(s) - R(s)] \]
\[ = H(s)[X(s) - G(s)Y(s)] \]

\[ \frac{Y(s)}{X(s)} = Q(s) = \frac{H(s)}{1 + G(s)H(s)} \]
Section 11.1 Linear Feedback Systems

• From the diagram, we obtain the relation

\[ Y(z) = H(z)E(z) \]
\[ E(z) = X(z) - R(z) \]
\[ R(z) = G(z)Y(z) \]

• These give the closed-loop system function is
Section 11.2 Some applications and consequences of feedback

- Section 11.2.1 Inverse System Design
  - Consider a continuous-time LTI system $P(s)$ and is configured as the figure
  - The closed-loop system function is
    \[
    \frac{Y(s)}{X(s)} = \frac{K}{1 + KP(s)}
    \]
  - If the gain $K$ is sufficiently large so that $KP(s) \gg 1$, then
    \[
    \frac{Y(s)}{X(s)} \approx \frac{1}{P(s)}
    \]
  - Applications: Operational amplifiers in feedback systems
    - The implementation of integrators by inserting a capacitor in the feedback path.
    - The implementation of logarithm of input by utilizing the exponential current-voltage characteristics of a diode in feedback path.
Section 11.2 Some applications and consequences of feedback

• Section 11.2.2 Compensation for non-ideal elements
  – Consider an open-loop frequency response $H(j\omega)$ which provides amplification over the specified frequency band, but which is not constant over that range.

  – The closed-loop system function is
    $$\frac{Y(j\omega)}{X(j\omega)} = \frac{H(j\omega)}{1 + KH(j\omega)}$$

  – If, over the specified frequency range, $|KH(j\omega)| >> 1$, then
    $$\frac{Y(s)}{X(s)} \approx \frac{1}{K}$$

  – Application:
    • Extending the bandwidth of an amplifier
      – Gain x Bandwidth = constant
Section 11.2 Some applications and consequences of feedback

- **Section 11.2.3 Stabilization of Unstable Systems**
  - Consider a first-order unstable system with
    \[
    H(s) = \frac{b}{s - a}, \quad a > 0
    \]
  - The closed-loop system function is
    \[
    \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + KH(s)} = \frac{b}{s - a + Kb}
    \]
  - The closed-loop system will be stable if the pole is moved into the left half of the \(s\)-plane. This gives
    \[
    Kb > a
    \]
Section 11.2 Some applications and consequences of feedback

- Section 11.2.3 Stabilization of Unstable Systems
  - Consider a second-order unstable system with
    \[ H(s) = \frac{b}{s^2 + a}, \quad a < 0 \]
    \[ x(t) \rightarrow + \rightarrow H(s) \rightarrow y(t) \]
  - The closed-loop system function is
    \[ \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + (K_1 + K_2s)H(s)} = \frac{b}{s^2 + bK_2s + (a + K_1b)} \]
  - The closed-loop system will be stable if the pole is moved into the left half of the \( s \)-plane. This gives
    \[ bK_2 > 0, \quad a + bK_1 > 0 \]

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Section 11.2 Some applications and consequences of feedback

- Section 11.2.3 Stabilization of Unstable Systems
  - Consider a first-order causal but unstable discrete-time system with
    
    \[ H(z) = \frac{1}{1 - 2z^{-1}} \]

  - The closed-loop system function is
    
    \[ \frac{Y(z)}{X(z)} = \frac{H(z)}{1 + 2\beta z^{-1}H(z)} = \frac{1}{1 - 2(1 - \beta)z^{-1}} \]

  - The closed-loop system will be stable if the pole is moved inside the unit circle.
    
    \[ |2(1 - \beta)| < 1 \]
    
    \[ \Rightarrow -1 < 2(1 - \beta) < 1 \]
    
    \[ \frac{1}{2} < \beta < \frac{3}{2} \]
Section 11.2 Some applications and consequences of feedback

- Section 11.2.4 Sampled-data feedback systems

\[ p[n] = y(nT) \quad x(t) = d[n] \quad \text{for} \quad nT \leq t < (n+1)T \]

\[ z(t) = d[n] \quad \text{for} \quad nT \leq t < (n+1)T \]

Figure 11.6 (a) A sampled-data feedback system using a zero-order hold; (b) equivalent discrete-time system.
Section 11.2 Some applications and consequences of feedback

- **Section 11.2.5 Tracking systems**
  - The output of system can be determined by
    
    \[ Y(z) = \frac{H_c(z)H_p(z)}{1 + H_c(z)H_p(z)} X(z) \]

    - Also, since \( E(z) = Y(z) - X(z) \)
    - This gives
      
      \[ E(z) = \frac{X(z)}{1 + H_c(z)H_p(z)} \]

    - Alternative, \( Y(z) = H_c(z)H_p(z)E(z) \)
      
      \[ E(z) = \frac{Y(z)}{H_c(z)H_p(z)} = \frac{X(z)}{1 + H_c(z)H_p(z)} \]
Section 11.2 Some applications and consequences of feedback

- Section 11.2.5 Tracking systems

  - For a good tracking performance, we would like to have a small error (all the time).

\[ e[n] \approx 0 \Rightarrow E(e^{j\omega}) = \frac{X(e^{j\omega})}{1 + H_c(e^{j\omega})H_p(e^{j\omega})} \approx 0 \]

  - This leads \( H_c(e^{j\omega})H_p(e^{j\omega}) \gg 1 \)

  Closed loop system may have trade-offs:
  - the damping ratio is too small
  - may become unstable

  - In some applications, we are interested in steady state situation.

\[ y[n] \to x[n] \quad for \quad n \to \infty \Rightarrow e[n] \to 0 \]
Section 11.2 Some applications and consequences of feedback

- Section 11.2.5 Tracking systems with disturbance
  - Example: with a disturbance $d[n]$ in the feedback path
  - The output of system can be determined by superposition.

- When disturbance $d[n] = 0$,
  \[ Y_1(z) = \]

- When input $x[n] = 0$,
  \[ Y_2(z) = \]
Section 11.2 Some applications and consequences of feedback

- Section 11.2.5 Tracking systems with disturbance
  - Example: with a disturbance $d[n]$ in the feedback path
  - The output of the system is then

\[
Y(z) = \frac{H_c(z)H_p(z)}{1 + H_c(z)H_p(z)} X(z) - \frac{H_c(z)H_p(z)}{1 + H_c(z)H_p(z)} D(z)
\]

- We aim to track the system with input and output signals, the tracking error is defined as

\[
E(z) = Y(z) - X(z)
\]

\[
= \frac{1}{1 + H_c(z)H_p(z)} X(z) - \frac{H_c(z)H_p(z)}{1 + H_c(z)H_p(z)} D(z)
\]

Main source of steady state error
Section 11.2 Some applications and consequences of feedback

- In general, several typical goals of tracking error
  - Minimize the error due to the disturbance with some designed parameters
    - For instance,
      \[
      \min_{\text{designed parameters}} \sum_{n=-\infty}^{\infty} e_d^2[n] = \min_{\text{designed parameters}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{H_c(e^{j\omega})H_p(e^{j\omega})D(e^{j\omega})}{1 + H_c(e^{j\omega})H_p(e^{j\omega})} \right|^2 d\omega
      \]

  - Minimize the error of overall system with some designed parameters
    - For instance,
      \[
      \min_{\text{designed parameters}} \sum_{n=-\infty}^{\infty} e^2[n] = \min_{\text{designed parameters}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{X(e^{j\omega}) - H_c(e^{j\omega})H_p(e^{j\omega})D(e^{j\omega})}{1 + H_c(e^{j\omega})H_p(e^{j\omega})} \right|^2 d\omega
      \]

  - Minimize the steady state error of system with some designed parameters
    - For instance,
      \[
      \min_{\text{designed parameters}} \lim_{n \to \infty} e[n] = \min_{\text{designed parameters}} \lim_{z \to 1} \{Y(z) - X(z)\}
      \]
Section 11.2 Some applications and consequences of feedback

- Section 11.2.6 Destabilization caused by feedback
  - Destabilizing effect of feedback is feedback in audio systems
  - The closed-loop system function is
    \[
    \frac{Y(s)}{X(s)} = \frac{K_1}{1 - K_1K_2e^{-sT}}
    \]
    pole:

  - The system is unstable if \( K_1K_2 > 1 \)
  - Same result would be obtained by using a technique that in Section 11.3
Section 11.3 Root-locus analysis of linear feedback systems

- From the applications we have discussed, a useful type of feedback system is that in which has an adjustable gain $K$ associated with.

- As this gain is varied, it is of interest to examine how the poles of the closed-loop system changes.
  - Broadening the bandwidth (Section 11.2.2)
  - Stabilizing an unstable system (Section 11.2.3)
  - Relocating the poles to improve system performance (Section 11.2.6)
Section 11.3 Root-locus analysis of linear feedback systems

• In this section, we discuss a particular method for examining the locus (i.e. the path) in the complex plane of the poles of the closed-loop system as an adjustable gain is varied.

• The procedure, referred to as the root-locus method, is a graphical technique for plotting the closed-loop poles of a rational system function as an adjustable gain is varied.

• The technique works in an identical manner of both continuous-time and discrete-time systems
Section 11.3 Root-locus analysis of linear feedback systems

- Example: Reexamine the discrete-time system, we have

\[ H(z) = \frac{1}{1-2z^{-1}} = \frac{z}{z-2} \quad G(z) = 2\beta z^{-1} = \frac{2\beta}{z} \]

- The closed-loop system function is

\[ \frac{Y(z)}{X(z)} = \frac{1}{1-2(1-\beta)z^{-1}} = \frac{z}{z-2(1-\beta)} \]

- The closed-loop zero is
- The closed-loop pole is

- The system is stable if
Section 11.3 Root-locus analysis of linear feedback systems

• Example: Consider a continuous-time system with

\[ H(s) = \frac{s}{s-2} \quad G(s) = \frac{2\beta}{s} \]

• The closed-loop system function is

\[ \frac{Y(s)}{X(s)} = \frac{s}{s-2(1-\beta)} \]

• The closed-loop zero is 0 and the closed-loop pole is \( 2(1-\beta) \)

• It is obvious that the locus of the pole as a function of \( \beta \) will be identical to the locus in previous example for discrete-time system.

• The system is stable if
Section 11.3 Root-locus analysis of linear feedback systems

• The location of poles are the solutions of the following equations

  Continuous-Time LTI Feedback System  Discrete-Time LTI Feedback System
  \[ 1 + KG(s)H(s) = 0 \]  \[ 1 + KG(z)H(z) = 0 \]

• For a complex system, it is possible to sketch accurately the locus of the poles as the value of gain parameter \( K \) is varied from \(-\infty\) to \( \infty \), without actually solving for the location of poles for any specific value of the gain.
Section 11.3 Root-locus analysis of linear feedback systems

• Usually, we can analyze the root locus with two different regions
  – Root-locus for $K$ is varied from $-\infty$ to 0
  – Root-locus for $K$ is varied from 0 to $\infty$

• We will phrase our discussion in terms of the Laplace transform variable $s$, with the understanding that it applies equally well to the discrete-time case.
Section 11.3 Root-locus analysis of linear feedback systems

- Consider a modification of the basic feedback system where either $G(s)$ or $H(s)$ is cascaded with an adjustable gain $K$ (real number).

- In either of these cases, the poles of the closed-loop system function is satisfied

$$1 + KG(s)H(s) = 0$$

- Rewrite the equation,

$$G(s)H(s) = \frac{-1}{K}$$

- The technique for plotting the root locus is based on the prosperities of this equation and its solutions.
Section 11.3 Root-locus analysis of linear feedback systems  
(The End Points of the Roots Locus)

• Criteria 1: End points of the root locus 
  – The closed-loop poles \( s_0 \) for \( K = 0 \), and \(|K| = + \infty\)
  – For \( K = 0 \), \( G(s_0)H(s_0) = \infty \), \( \Rightarrow \) closed-loop poles \( s_0 = \) poles of \( G(s)H(s) \)
  – For \( |K| = \infty \), \( G(s_0)H(s_0) = 0 \) \( \Rightarrow \) closed-loop poles \( s_0 = \) zeroes of \( G(s)H(s) \)
  – If the order of the numerator of \( G(s)H(s) \) is smaller than that of the denominator, then some of zeros, equal in number to the difference in order between the denominator and numerator, will be at infinity.

\[
G(s)H(s) = \frac{-1}{K}
\]
Section 11.3 Root-locus analysis of linear feedback systems  
(The Angle Criterion of the Roots Locus)

• Criteria 2: The Angle Criterion for the closed-loop pole $s_0$
  – From $G(s_0)H(s_0) = \frac{-1}{K}$, $G(s_0)H(s_0)$ must be real.

  – The magnitude criterion at closed-loop pole $s_0$,
    \[ |G(s_0)H(s_0)| = \frac{1}{K} = \begin{cases} 
    \frac{1}{K}, & K > 0 \\
    -\frac{1}{K}, & K < 0 
  \end{cases} \]
    The value of $K$ can be evaluated by lengths of Vectors.

  – The angle criterion at closed-loop pole $s_0$,
    \[ \angle G(s_0)H(s_0) = q\pi, \quad q \text{ is an integer} \]

  – In summary, $s_0$ is the closed-loop pole
    \[ \text{if} \quad K > 0, \quad G(s_0)H(s_0) < 0 \Rightarrow \angle G(s_0)H(s_0) = q\pi, \quad q \text{ is an odd integer} \]
    \[ \text{if} \quad K < 0, \quad G(s_0)H(s_0) > 0 \Rightarrow \angle G(s_0)H(s_0) = q\pi, \quad q \text{ is an even integer} \]
Section 11.3 Root-locus analysis of linear feedback systems

- **Example 11.1** Consider \( G(s)H(s) = \frac{1}{(s + 1)(s + 2)} \)

- Examine a point \( s_0 \) whether it is a closed-loop pole of closed loop system.
  
  \[
  G(s_0)H(s_0) = \frac{1}{(s_0 + 1)(s_0 + 2)} = \frac{-1}{K}
  \]

- Consider a point \( s_0 \) that lies on real axis of the \( s \)-plane
  
  - Case 1: \( s_0 \) is real and \( s_0 > -1 \),
  
  \[
  \angle G(s_0)H(s_0) = 0 = 0 \cdot \pi \]
  
  It is a closed-loop pole. 
  
  \( \Rightarrow \) Lies on the roots locus with \( K < 0 \)
Section 11.3 Root-locus analysis of linear feedback systems

- Example 11.1 (Cont’d)
- Consider a point $s_0$ that lies on real axis of the $s$-plane
  - Case 2: $s_0$ is real and $-2 < s_0 < -1$,

$$\angle G(s_0)H(s_0) = -\pi$$

It is a closed-loop pole.
⇒ Lies on the roots locus with $K > 0$
Section 11.3 Root-locus analysis of linear feedback systems

- Example 11.1 (Cont’d)
- Consider a point \( s_0 \) that lies on real axis of the \( s \)-plane
  - Case 3: \( s_0 \) is real and \( s_0 < -2 \),
    \[
    \angle G(s_0)H(s_0) = -2\pi
    \]
    It is a closed-loop pole.
    \Rightarrow \text{Lies on the roots locus with } K < 0
Section 11.3 Root-locus analysis of linear feedback systems

- **Example 11.1 (Cont’d)**
- Consider a point $s_0$ that lies on complex plane
  - **Case 4:** $s_0$ is complex point in the upper half plane,
    \[ \angle G(s_0)H(s_0) = -(\theta + \phi) \]
  - As $0 < \theta < \pi$ and $0 < \phi < \pi$, thus $-2\pi < \theta + \phi < 0$
  - There is no point in the upper half-plane can be the locus for $K < 0$ as $\angle G(s_0)H(s_0)$ never equals an even multiple of $\pi$.
  - In addition, if $s_0$ is be on the locus for $K > 0$, we must have $\angle G(s_0)H(s_0) = -(\theta + \phi) = -\pi$
Section 11.3 Root-locus analysis of linear feedback systems
(Properties of the Root Locus)

• Property 1: For $K = 0$, the solution of $G(s)H(s) = 1/K$ are the poles of $G(s)H(s)$. Since we are assuming $n$ poles, the root locus has $n$ branches, each one starting (for $K=0$) at a pole of $G(s)H(s)$. 

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Section 11.3 Root-locus analysis of linear feedback systems
(Properties of the Root Locus)

- Property 2: As $|K| \to \infty$, each branch of the root locus approaches a zero of $G(s)H(s)$. Since we are assuming that $m \leq n$, $n - m$ of these zeros are at infinity.

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Section 11.3 Root-locus analysis of linear feedback systems
(Properties of the Root Locus)

- Property 3: Parts of the real $s$-axis that lie to the left of an odd number real poles and zeros of $G(s)H(s)$ are on the root locus for $K > 0$. Parts of the real $s$-axis that lie to the left of an even number (possibly zero) real poles and zeros of $G(s)H(s)$ are on the root locus for $K < 0$. 
Section 11.3 Root-locus analysis of linear feedback systems
(Properties of the Root Locus)

• Property 4: Branches of the root locus between two real poles must break off into the complex plane for $|K|$ is large enough.
Section 11.3 Root-locus analysis of linear feedback systems

- Example 11.2 Consider a continuous-time feedback system with

\[ G(s)H(s) = \frac{s-1}{(s+1)(s+2)} \]

- From Properties 1 and 2, the root locus for either \( K \) positive or \( K \) negative starts at the points \( s = -1 \) and \( s = -2 \). One branch terminates at the points \( s = 1 \) and the other at infinity.
- By property 3,
  - For \( K > 0 \), the branches are \( s < -2 \) and \( -1 < s < 1 \)
- For \( K > 0 \), if \( K \) is sufficient large, the system becomes unstable.
  - The root locus passes through \( s = 0 \)

\[ K = \left. \frac{1}{G(s)H(s)} \right|_{s=0} = \]
Example 11.2 (Cont’d)

By property 3, for $K < 0$, the branches are $-2 < s < -1$ and $s > 1$

Property 4, the branches break off in between $-2 < s < -1$

For $K < 0$, if $K$ is sufficient large in magnitude, the system becomes unstable.

- The root locus passes through $s = j2.2361$
  
  \[
  K = -\left| \frac{1}{G(s)H(s)} \right|_{s=j2.2361} = -3
  \]

- With the use of MATLAB command
  
  \texttt{[k, poles]=rlocfind(sys); (just point and click on any desired point on the plot after entering this command)}

We have $K = -3$
Section 11.3 Root-locus analysis of linear feedback systems

- Example 11.2 (Cont’d)
- We can also find the roots of imaginary axis mathematically.
- The root locus passes through $s = j\omega$ such that $\angle G(j\omega)H(j\omega) = 0$

\[
\left(\pi - \tan^{-1} \omega\right) - \tan^{-1} \omega - \tan^{-1} \frac{\omega}{2} = 0
\]

\[
\tan\left(2 \tan^{-1} \omega\right) = \tan\left(\pi - \tan^{-1} \frac{\omega}{2}\right)
\]

\[
\frac{2\omega}{1-\omega^2} = -\frac{\omega}{2}
\]

\[
\omega = \pm \sqrt{5} = \pm 2.2361 \quad \text{or} \quad 0
\]

- This gives

\[
K = -\left. \frac{1}{G(s)H(s)} \right|_{s=j2.2361} = -3
\]
Example 11.3 Consider a discrete-time feedback system with

\[ G(z)H(z) = \frac{z^{-1}}{(1 - \frac{1}{2} z^-)(1 - \frac{1}{4} z^-)} = \frac{z}{(z - \frac{1}{2})(z - \frac{1}{4})} \]

For \( K > 0 \), we see that the transition from stability to instability occurs when one of the closed-loop poles is at \( z = -1 \).

For \( K < 0 \), we see that the transition from stability to instability occurs when one of the closed-loop poles is at \( z = 1 \).
Section 11.3 Root-locus analysis of linear feedback systems

- Other properties of the root locus
  - The root locus is symmetric with respect to the real axis.
  - As $|K| \to \infty$, $n - m$ each branch of the root locus approaches a zero of asymptotically $n - m$ straight lines (called asymptotes) with angles
    \[
    \theta = \frac{q \pi}{n - m}, \quad \begin{cases} 
    q = \text{odd integer}, & K > 0 \\
    q = \text{even integer}, & K < 0
    \end{cases}
    \]
    and the starting point of all asymptotes is on the real axis at
    \[
    \kappa = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n - m} = \frac{\text{poles} - \text{zeros}}{n - m}
    \]
  - The breakaway and breakin points are among the roots of equation
    \[
    \frac{d[G(s)H(s)]}{ds} = 0
    \]
Section 11.3 Root-locus analysis of linear feedback systems

- Other properties of the root locus
  - The angle of departure $\phi_k$ from the pole $p_k$ with order $l$ is given by
    \[
    \phi_k = \frac{\sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{j=1, j\neq k}^{n} \angle(p_k - p_j) \pm q\pi}{l}
    \]
    where $q = 0$ for $K < 0$ and $q = 1$ for $K > 0$
  - The angle of arrival $\psi_k$ at the zero $z_k$ with order $l$ is given by
    \[
    \psi_k = \frac{-\sum_{i=1, i\neq k}^{m} \angle(z_k - z_i) + \sum_{j=1}^{n} \angle(z_k - p_j) \pm q\pi}{l}
    \]
    where $q = 0$ for $K < 0$ and $q = 1$ for $K > 0
Section 11.3 Root-locus analysis of linear feedback systems

- Sketch the root locus of a continuous-time feedback system with

\[ G(s)H(s) = \frac{(s^2 + 4s + 8)}{s(s + 4)(s + 5)(s^2 + 2s + 2)} \]

and \( K > 0 \).
- poles: 0, -4, -5, -1 + j and -1 - j  
- zeros: -2 + 2 j and -2 – 2 j
- From properties 1 and 2, three branches terminate at infinity.
- Property 3, the root locus on the real axis are shown in red color.
Section 11.3 Root-locus analysis of linear feedback systems

- The branches break off in between \(-4 < s < 0\) with

\[
\frac{d}{ds} G(s)H(s) = \frac{d}{ds} \frac{N(s)}{D(s)} = \frac{D(s) \frac{dN(s)}{ds} - N(s) \frac{dD(s)}{ds}}{D^2(s)} = 0
\]

```
>> num=[1 4 8]; % vector for numerator of G(s)H(s)
den=conv(poly([0 -4 -5]), [1 2 1]); % vector for denominator of G(s)H(s)
diff_num=num(1:2).*[2 1]; % vector for differentiation of numerator of G(s)H(s)
diff_den=den(1:5).*[5:-1:1]; % vector for differentiation of denominator of G(s)H(s)
new_Num=conv(den, diff_num) - conv(num, diff_den); % vector for new numerator of d[G(s)H(s)]/ds
roots(new_Num) % the roots of the equation d[G(s)H(s)]/ds = 0
```

-4.5323
-2.1050 + 2.9299i
-2.1050 - 2.9299i
-3.5149
-0.7038 + 0.4728i
-0.7038 - 0.4728i
Section 11.3 Root-locus analysis of linear feedback systems

- Angle of arrival at the zero -2 + 2 j

\[
\psi_k = -\sum_{i=1,i\neq k}^{m} \angle(z_k - z_i) + \sum_{j=1}^{n} \angle(z_k - p_j) \pm q\pi
\]

\[
= -\frac{\pi}{2} + \left( \frac{3\pi}{4} + \frac{3\pi}{4} + \pi - \tan^{-1} \frac{3}{4} + \frac{\pi}{3} + \tan^{-1} \frac{2}{3} \right) - \pi
\]

\[
= 1.0396\pi
\]

- Angle of departure at the pole -1 + j

\[
\phi_k = \sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{j=1,j\neq k}^{n} \angle(p_k - p_j) \pm q\pi
\]

\[
= \left( -\frac{\pi}{4} + \tan^{-1} \frac{3}{4} \right) - \left( \frac{3\pi}{4} + \frac{\pi}{2} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{4} \right) + \pi
\]

\[
= -0.2828\pi
\]
Section 11.3 Root-locus analysis of linear feedback systems

- The 3 asymptotes with angles $\theta = \frac{q\pi}{3} = \left\{ -\frac{\pi}{3}, \frac{\pi}{3}, \pi \right\}$
- Starting point of all asymptotes is
  
  $\kappa = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n - m} = \frac{(0 - 4 - 5 - 1 + j - 1 - j) - ( -2 + j - 2 - j)}{3} = -\frac{7}{3}$
Section 11.4 The Nyquist Stability Criterion

- In this section, another method for determination of the stability of feedback systems as a function of an adjustable gain. This technique is called Nyquist Stability Criterion.
- Comparison of root locus method and Nyquist stability criterion.

<table>
<thead>
<tr>
<th></th>
<th>Root Locus Method</th>
<th>Nyquist Stability Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location of the closed-loop poles as a function of $K$</td>
<td>Provide detailed information</td>
<td>Does not provide detailed information</td>
</tr>
<tr>
<td>System functions</td>
<td>Rational</td>
<td>Both Rational or irrational</td>
</tr>
<tr>
<td>Analytic description of system functions of forward and feedback paths</td>
<td>Required</td>
<td>Not required</td>
</tr>
</tbody>
</table>
Section 11.4 The Nyquist Stability Criterion

• Recall the poles of closed-loop systems are the solutions of the equations
  
  Continuous-Time LTI Feedback System  \[ 1 + KG(s)H(s) = 0 \]
  
  Discrete-Time LTI Feedback System  \[ 1 + KG(z)H(z) = 0 \]

• The Nyquist criterion fixes the stability by examination of the values of \( G(s)H(s) \) along \( j\omega \)-axis and the values of \( G(z)H(z) \) along the unit circle.

• Before examining these, the basic concept of the encirclement property will be discussed.
Section 11.4.1 The Encirclement Property

• Consider a general rational function $W(p)$, where $p$ is a complex variable.
  - For continuous-time feedback system, $p = s$
  - For discrete-time feedback system, $p = z$

• Suppose that we plot $W(p)$ for values $p$ of a closed contour $C$ in the $p$-plane, which we traverse in a clockwise direction.

![Diagram showing the encirclement property](image-url)
Section 11.4.1 The Encirclement Property

• Let us illustrate the encirclement property through an example. Consider a function

\[ W(p) = (p + a_1)(p + a_2) \quad \text{where} \quad a_1, a_2 < 0 \]

• Take a closed contour \( C \) in the \( p \)-plane. The magnitude and phase of the function are

\[ |W(p)| = |p + a_1||p + a_2| \]

\[ \angle W(p) = \angle(p + a_1) + \angle(p + a_2) = \phi_1 + \phi_2 \]
Section 11.4.1 The Encirclement Property

• Consider the net change of angle along the contour $C$,

$$\oint \angle W(p) \, dp = \oint \phi_1 \, dp + \oint \phi_2 \, dp = (-2\pi) + 0$$

• That’s, the plot $W(p)$ encircles the origin once in the clockwise direction.
Section 11.4.1 The Encirclement Property

• For an arbitrary rational $W(p)$, as we traverse a closed contour in the clockwise direction,
  - any poles and zeros of $W(p)$ outside the contour $C$
    • Net change of angle of $W(p) = \pi$
  - each zero of $W(p)$ inside the contour $C$
    • Net change of angle of $W(p) = -\pi$
  - each pole of $W(p)$ inside the contour $C$
    • Net change of angle of $W(p) = \pi$

• Each net change of $-2\pi$ in $W(p)$ corresponding to one clockwise encirclement of the origin in the plot of $W(p)$. 
Section 11.4.1 The Encirclement Property

• Encirclement Property
  – As a closed contour $C$ in the p-plane is traversed once in the clockwise direction, the corresponding plot of $W(p)$ for values of $p$ along the contour encircles the origin in the clockwise direction a net number of times equal to the number of zeros minus the number of poles contained within the contour.

• Does $W(p)$ arise with any regions of convergence as the Laplace or z-transform of any signal?
Section 11.4.1 The Encirclement Property

• Example 11.4 Consider the function $W(p) = \frac{p - 1}{(p + 1)(p^2 + p + 1)}$
The Nyquist Criterion for Continuous-Time LTI feedback systems

- Stability of continuous-time LTI feedback system requires that no zeros of $1 + KG(s)H(s)$, or equivalent, of the function

$$R(s) = \frac{1}{K} + G(s)H(s)$$

lie in the right half of $s$-plane.

- Consider the contour $C$,

- By encirclement property, as traverses the contour $C$, we have

The number of clockwise encirclements of the origin of the plot $R(s)$

= The number of zeros minus the number of poles of $R(s)$ in the right half of s-plane
Section 11.4.2 The Nyquist Criterion for Continuous-Time LTI feedback systems

- Interesting things from

\[ R(s) = \frac{1}{K} + G(s)H(s) = \frac{1+KG(s)H(s)}{K} \]

- The zeros of \( R(s) \) = the roots of \([ 1 + KG(s)H(s) ]\) = The closed-loop poles of the system

- The poles of \( R(s) \) = The poles of \( G(s)H(s) \)

- With \( G(s) = \frac{N_G(s)}{D_G(s)} \), \( H(s) = \frac{N_H(s)}{D_H(s)} \)

\[ R(s) = \frac{1+KG(s)H(s)}{K} = \]
Section 11.4.2 The Nyquist Criterion for Continuous-Time LTI feedback systems

• In addition, \( G(j\omega)H(j\omega) = R(j\omega) - 1/K \). (Shift the curve of \( R(j\omega) \))

• From the encirclement property, we have

  The number of clockwise encirclements of the origin of the plot \( R(s) \) = The number of zeros of \( R(s) \) minus the number of poles of \( R(s) \) in the right half of \( s \)-plane

New center

The number of clockwise encirclements of the point \(-1/K\) by the Nyquist plot = The number of right-half plane closed-loop poles minus the number of right-half plane poles of \( G(s)H(s) \).

Note: The plot of \( G(j\omega)H(j\omega) \) as \( \omega \) varies from \(-\infty\) to \(+\infty\) is called the Nyquist plot.
Section 11.4.2 The Nyquist Criterion for Continuous-Time LTI feedback systems

- While the open-loop system $G(s)H(s)$ may have unstable poles.
- For the closed-loop system to be stable, we require no right-half plane closed-loop poles.
- This yields the continuous-time Nyquist stability criterion:

  The number of clockwise encirclements of the point $-1/K$ by the Nyquist plot
  minus the number of right-half plane poles of $G(s)H(s)$.

- In summary,

  Continuous-Time Nyquist Stability Criterion:
  For the closed-loop system to be stable, the net number of counterclockwise encirclements of the point $-1/K$ by the Nyquist plot of $G(j\omega)H(j\omega)$ as $\omega$ varies from $-\infty$ to $+\infty$ must equal to the number of right-half plane poles of $G(s)H(s)$. 

This is the information we need.
Section 11.4.2 The Nyquist Criterion for Continuous-Time LTI feedback systems

• Example 11.5 An example of rational and stable function. Let

\[ G(s)H(s) = \frac{1}{(s+1)(\frac{1}{2}s+1)} \]

• There are no right-half plane open-loop poles of \( G(s)H(s) \), the Nyquist criterion requires that, for stability, there are no net encirclements of the point \(-1 / K\).
Example 11.6  An example of rational but unstable function. Let

\[ G(s)H(s) = \frac{s + 1}{(s - 1)(\frac{1}{2}s + 1)} \]

There are one right-half plane open-loop poles of \( G(s)H(s) \), the Nyquist criterion requires that, for stability, we requires one counterclockwise encirclements of the point \(-1 / K\).
Section 11.4.2 The Nyquist Criterion for Continuous-Time LTI feedback systems

• Example 11.7 An example of irrational function: the acoustic feedback system discussed in Section 11.2.6. Let \( K = K_1 K_2 \)

\[
G(s)H(s) = -e^{-sT} = e^{-(sT+j\pi)}
\]

• Is this open-loop system function stable?

\[
G(j\omega)H(j\omega) = -e^{-sT} = e^{-j(\omega T+\pi)}
\]

• There are no right-half plane open-loop poles of \( G(s)H(s) \), the Nyquist criterion requires that, for stability, there are no net encirclements of the point \(-1/K\).

\[ -\frac{1}{K} < -1 \quad \text{or} \quad -\frac{1}{K} > 1 \Rightarrow -1 < K < 1 \]

• Since \( K_1 \) and \( K_2 \) represent an acoustic gain and attenuation, respectively, they are both positive, which yields the stability criterion \( K_1 K_2 < 1 \)
Section 11.4.2 The Nyquist Criterion for Discrete-Time LTI feedback systems

- As the Nyquist Criterion can be applied onto to both $s$- and $z$-plane, the discrete-time Nyquist Stability criterion is closely parallel those of the continuous-time.

Discrete-Time Nyquist Stability Criterion:
For the closed-loop system to be stable, the net number of counterclockwise encirclements of the point $-1/K$ by the Nyquist plot of $G(e^{j\omega})H(e^{j\omega})$ as $\omega$ varies from 0 to $2\pi$ must equal to the number of poles of $G(z)H(z)$ outside the unit circle.
Section 11.4.2 The Nyquist Criterion for Discrete-Time LTI feedback systems

- Example 11.8 An example of rational and stable function. Let

\[ G(z)H(z) = \frac{z^{-2}}{1 + \frac{1}{2} z^{-1}} = \frac{1}{z(z + \frac{1}{2})} \]

Figure 11.24

- There are no open-loop poles of \( G(z)H(z) \) outside the unit circle, the Nyquist criterion requires that, for stability, there are no net encirclements of the point \(-1 / K\).
Section 11.5 Gain and Phase Margins

• In this section, we introduce and examine the concept of the margin of stability in a feedback system.
• Assume a typical feedback system, as shown in Figure 11.25, is stable.

\[ \begin{align*}
&\text{Figure 11.25} \\
&\text{Figure 11.26}
\end{align*} \]

• To assess the margin of stability, we introduce two building blocks for possibility of a gain \( K \) and phase shift \( \phi \) in the feedback path as shown in Figure 11.26.
Section 11.5 Gain and Phase Margins

• Consider the equation for the poles of closed-loop system is

\[ D(K, \phi, s) = 1 + Ke^{-j\phi}G(s)H(s) = 0 \]

\[ D(K, \phi, \omega_0) = 1 + Ke^{-j\phi}G(j\omega_0)H(j\omega_0) = 0 \]

\[ D(1, \phi, \omega_0) = 1 + e^{-j\phi}G(j\omega_0)H(j\omega_0) = 0 \]

\[ D(K,0,\omega_1) = 1 + KG(j\omega_1)H(j\omega_1) = 0 \]

• Definition:
  – Gain Margin of the feedback system, \( K > 1 \)
    • The minimum amount of additional gain \( K \) with \( \phi = 0 \), that is required the closed-loop system becomes unstable.
  – Phase Margin of the feedback system, \( \phi > 0 \)
    • The minimum amount of additional phase \( \phi \) with \( K = 1 \), that is required the closed-loop system becomes unstable.
Section 11.5 Gain and Phase Margins

- Another viewing angle from the Nyquist plot: \( G(j\omega)H(j\omega) \)

\[
KG(j\omega_0)H(j\omega_0) = K|G(j\omega_0)H(j\omega_0)|e^{j\angle[G(j\omega_0)H(j\omega_0)]}
\]

Gain Margin:
\( KG(j\omega_0)H(j\omega_0) = -1 \)
\[ \Rightarrow \omega_0 \quad \text{such that} \quad \angle G(j\omega_0)H(j\omega_0) = \pi \]

\[
K = \frac{1}{|G(j\omega_0)H(j\omega_0)|}
\]
Section 11.5 Gain and Phase Margins

- Another viewing angle from the Nyquist plot: $G(j\omega)H(j\omega)$

\[ e^{-j\phi}G(j\omega_0)H(j\omega_0) = |G(j\omega_0)H(j\omega_0)|e^{j[\angle(G(j\omega_0)H(j\omega_0)]-\phi) \]

Phase Margin:

\[ e^{-j\phi}G(j\omega_0)H(j\omega_0) = -1 \]

\[ \Rightarrow \omega_0 \text{ such that } |G(j\omega_0)H(j\omega_0)| = 1 \]

\[ \phi = \angle[G(j\omega_0)H(j\omega_0)] - \pi \]
Section 11.5 Gain and Phase Margins

- Example 11.9 Consider a system function

\[ G(s)H(s) = \frac{4\left(\frac{1}{2}s + 1\right)}{s(2s + 1)[1 + 0.05s + (0.125s)^2]} \]

Figure 11.27 Use of Bode plots to calculate gain and phase margins for the system of Example 11.9.

Figure 11.28 Log magnitude-phase plot for the system of Example 11.9.
Section 11.5 Gain and Phase Margins

• Example 11.10 Consider a system function

\[ G(s)H(s) = \frac{1}{\tau s + 1}, \quad \tau > 0 \]

\( H(s) = \frac{1}{\tau s + 1} \)

Figure 11.29 Log magnitude-phase plot for the first-order system of Example 11.10.

Figure 11.30 (a) First-order feedback system with possible gain and phase variations in the feedback path; (b) root locus for this system with \( \phi = 0, K > 0 \).
Section 11.5 Gain and Phase Margins

• Example 11.11 Consider the system

\[ H(s) = \frac{1}{s^2 + s + 1}, \quad G(s) = 1 \]
Section 11.5 Gain and Phase Margins

- Consider another delayed system

\[ G(s) = e^{-s\tau} \]
Section 11.5 Gain and Phase Margins

- Example 11.12 Reexamine the acoustic feedback system.
- We assume the system has been designed with $K_1K_2 < 1$, so that the closed-loop system is stable.
Section 11.5 Gain and Phase Margins

• Example 11.13 Consider a discrete-time open-loop system

\[ G(z)H(z) = \frac{\frac{7\sqrt{2}}{4} z^{-1}}{1 - \frac{7\sqrt{2}}{8} z^{-1} + \frac{49}{64} z^{-2}} \]
Summary

• In this chapter, we have examined a number of the applications and several techniques for the analysis of feedback systems.
  – Applications of feedback systems
  – Root-locus method
  – Nyquist criterion
  – Gain and Phase margins
Linear Feedback Systems

• Readings
  – Section 11.0 Introduction
  – Section 11.1 Linear Feedback Systems
  – Section 11.2 Some applications and consequence of feedback
  – Section 11.3 Root-locus analysis of linear feedback systems
  – Section 11.4 The Nyquist stability criterion
  – Section 11.5 Gain and phase margins
Where we are

Continuous-time

Hardware Implementation ↔ Closed-loop Systems

Open-loop Systems → State-space model

Differential equations

System Characteristics

System Responses

CTFT

Laplace Transform

Discrete-time

Closed-loop Systems ↔ Hardware Implementation

Open-loop Systems

State-space model

Mapping

Difference equations

System Characteristics

System Responses

DTFT

z-Transform

ELEC 215: Tim Woo

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